

ON THE SCHRÖDINGER AND HEAT EQUATIONS FOR NONNEGATIVE POTENTIALS⁽¹⁾

BY
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1. **Introduction.** Consider the equation

$$(1) \quad \frac{1}{\sigma} \frac{\partial u(x, t)}{\partial t} = (\Delta - V(x))u(x, t)$$

x varying over Euclidean n -space, and $0 \leq t < \infty$, with the initial condition $u(x, 0+) = f(x)$. For positive σ , this is the heat equation; for purely imaginary σ , it is Schrödinger's equation for a particle in a force field. In his dissertation, and later in a published article [6], R. Feynman indicated how one might get this solution as a limit of averages over polygonal paths. His prescription was not mathematically rigorous, however, since it involved infinite constants and integration with respect to a fictitious translation-invariant measure in an infinite product of real lines. In the case of the heat equation, Kac [11] made this precise by using Wiener measure instead. The approximating averages became finite-dimensional approximants to a Wiener integral: for sufficiently well-behaved $V \geq 0$ and f ,

$$(2) \quad \begin{aligned} u(x, t) &= E \left\{ \exp \left(- \int_0^t V(\xi_s + x) ds \right) f(\xi_t + x) \right\} \\ &= \lim E \left\{ \exp \left[- \sum_i V(\xi_{s_i} + x) \Delta s_i \right] f(\xi_t + x) \right\} \end{aligned}$$

where ξ_t is Brownian motion with parameter σ , starting at 0, and the limit is taken as $\max \Delta s_i \rightarrow 0$. This was developed further by Rosenblatt [16] and Ray [15]. The problem was treated for larger classes of V and for more general Markov-processes by Gettoor [8; 9], Dynkin, Volkonskiĭ [17], et al. Gel'fand and Yaglom [7] indicated heuristically how the same sort of approximating finite-dimensional integrals might be used to get solutions to the Schrödinger equation. They made the error of stating that, for $\text{re } \sigma \neq 0$, the limit could be expressed as an integral over path space. Cameron [3] pointed out this error, but proved rigorously (for a rather narrow class of V : required to satisfy certain analyticity assumptions) using certain other approximating expressions, that the limit existed for $\text{re } \sigma > 0$.

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The case of purely imaginary σ was gotten as a boundary value of an analytic function. His approximants, incidentally, were not the same as those used by the previous authors; they corresponded to using a Simpson's rule rather than a Riemann sum to approximate $\int_0^t V(\xi_s)ds$. Recently, D. Babbitt, in his doctoral dissertation [1], noted that Feynman's program could be carried out rather effectively if one regarded t rather than σ as an analytic parameter. In this way, he defined a semigroup which, for $\text{re } \sigma > 0$, gave the solution to (1), and approximated it by an expression like (2). This worked for V satisfying a local Lipschitz condition (and, again, ≥ 0 ; or, more generally, bounded below).

In the present paper we proceed as follows. First, we construct a semigroup T_V^t which gives a solution to (1), for arbitrary positive measurable V . This was already done more generally by Gettoor [8; 9], so this is merely an exposition of a special case. Next, we investigate some smoothing properties of the operators T_V^t . In §4, an infinitesimal generator for T_V^t is shown to exist. By means of the generator, T_V^ζ is defined for $\text{re } \zeta \geq 0$. This is then approximated by Babbitt's method, but in a more general situation. In §6, it is shown that T_V^ζ may be obtained from a Green's function, whose regularity properties are investigated.

It should be added that we have just learned that E. Nelson [14] has also succeeded in constructing a semigroup and a Feynman approximation for a large class of potentials, not necessarily bounded below.

For general background and bibliographical references, we refer the reader to [2; 7; 12]. The author would like to express his gratitude to D. G. Babbitt for the opportunity of seeing his manuscript at an early stage; and to E. Nelson for a stimulating discussion, and in particular, for pointing out an error in an earlier version of Theorem 3.5.

2. Brownian motion in k -dimensions. Let $\mathcal{H}$ be a real Hilbert space of dimension k , the inner product being denoted by $x \cdot y$. Let Ω be the set of continuous functions $\omega: [0, \infty) \rightarrow \mathcal{H}$, and ξ_t the function from Ω to $\mathcal{H}$ defined by $\xi_t(\omega) = \omega(t)$. Let $\mathcal{F}_t$ be the smallest σ -field of subsets of Ω for which ξ_s is measurable for all s in $[0, t]$, and let $\mathcal{F}$ be the smallest σ -field containing all the $\mathcal{F}_t$.

Let

$$G_\sigma(x) = (\pi\sigma)^{-k/2} e^{-\|x\|^2/\sigma} \quad (\sigma > 0, x \in \mathcal{H}).$$

Observe that $G_\sigma(x/\sqrt{t_0})d(x/\sqrt{t_0}) = G_{t_0\sigma}(x)dx$. More generally, if L_0 is a non-singular linear transformation on $\mathcal{H}$ such that $L_0^*L_0 = t_0I$, then

$$G_\sigma(L_0x)d(L_0x) = G_{t_0\sigma}(x)dx.$$

There is a unique probability measure Pr_x^σ on $\mathcal{F}$ characterized by the property that if $h_1, \dots, h_n > 0$, and $t_j = h_1 + \dots + h_j$, and $S_0, \dots, S_n$ are Borel sets in $\mathcal{H}$, then $\text{Pr}_x^\sigma\{\xi_0 \in S_0, \xi_{t_1} \in S_1, \dots, \xi_{t_n} \in S_n\} = 1_{S_0}(x) \int_{S_1} \dots \int_{S_n} G_{h_1}(x_1 - x) \dots G_{h_n}(x_n - x_{n-1}) dx_1 \dots dx_n$.

(NOTATION. 1_S will mean the characteristic function of S . I_S will mean the operation of multiplication by 1_S .)

So Pr_x^σ makes ξ_t into a temporally homogeneous Markov process starting at x , with transition function

$$\text{Pr}_x^\sigma\{\xi_t = dy \mid \xi_{t_0} = y_0\} = G_{\sigma(t-t_0)}(y-y_0)dy.$$

When $\sigma = 1$ the superscript in Pr_x^σ will often be omitted, and when $x = 0$ the subscript will often be omitted. We also denote by $E_x\{\dots\}$ the operation of integration with respect to Pr_x^σ , and make the same conventions about omitting σ when it equals 1 and x when it equals 0.

Let us introduce some transformations on Ω :

(1) if $t > 0$, set $j_t(\omega)(s) = \omega(st)$. Thus,

$$\xi_s \cdot j_t = \xi_{st},$$

(2) if L is a continuous map: $\mathcal{K} \rightarrow \mathcal{K}$, we set $k_L(\omega)(s) = L\omega(s)$. Thus,

$$\xi_s(k_L(\omega)) = L\xi_s(\omega).$$

REMARK 2.1. If $L y = L_0 y - x_0$, where L_0 is a linear transformation on $\mathcal{K}$ with $L_0^* L_0 = t_0^{-1} I$, then:

$$\text{Pr}_x^\sigma \cdot k_L^{-1} = \text{Pr}_{L_0^{-1}x}^{t_0\sigma}.$$

In particular:

$$\text{Pr}^\sigma \cdot L^{-1} = \text{Pr}_{x_0}^{t_0\sigma}.$$

Proof. This merely involves computing, for both sides of the equation, the measure of the set where $\xi_0 \in S_0$, $\xi_{h_1} \in S_1, \dots, \xi_{h_1+\dots+h_n} \in S_n$, and using the transformation property of G_σ observed above. The details are omitted.

3. The semigroup obtained from a potential V . Call a complex Borel measurable function f on $\mathcal{K}$ *moderate* if $\int |f(x)| e^{-c\|x\|^2} dx < \infty$ for each $c > 0$. These functions form a translation-invariant linear space $\mathcal{M}$. Observe also that a moderate f is Lebesgue integrable on compact sets. For any $t > 0$ and positive measurable V (the value $+\infty$ being permitted), we set

$$\begin{aligned} T_V^t f(x) &= E_x \left\{ \exp \left[- \int_0^t V(\xi_s) ds \right] f(\xi_t) \right\} \\ &= E \left\{ \exp \left[- \int_0^t V(\xi_s + x) ds \right] f(\xi_t + x) \right\}. \end{aligned}$$

This makes sense, since

$$\left| \exp \left[- \int_0^t V(\xi_s) ds \right] f(\xi_t) \right| \leq |f(\xi_t)|,$$

and

$$E_x\{|f(\xi_t)|\} \leq (\pi t)^{-k/2} \int |f(y)| e^{-\|y-x\|^2/t} dy.$$

Now,

$$\frac{\|y-x\|^2}{t} = \frac{\|y/2 - 2x\|^2}{t} - \frac{3\|y\|^2}{4t} + \frac{3\|x\|^2}{t},$$

so

$$e^{-\|y-x\|^2/t} \leq e^{-3\|y\|^2/4t} e^{3\|x\|^2/t},$$

and

$$E_x\{|f(\xi_t)|\} \leq (\pi c)^{-k/2} e^{-3\|x\|^2/t} \int |f(y)| e^{-\|y\|^2/c} dy,$$

where $c = (3/4)t$.

Measurability of the integrand is not difficult to see, by first considering V of the form 1_S , S open, and then approximating. Or see [8; 9] for proof.

THEOREM 3.1. T_V^t is a linear transformation from $\mathcal{M}$ to $\mathcal{M}$, sending a.e. nonnegative functions to nonnegative functions. If $0 \leq f_n \uparrow f$ a.e., then $T_V^t f_n \uparrow T_V^t f$ a.e. Finally, $T_V^{s+t} = T_V^s T_V^t$.

Proof. All statements but the first and last are evident. To prove these:

$$\begin{aligned} \int |T_V^t f(x)| e^{-c\|x\|^2} dx &\leq \int E_x\{|f(\xi_t)|\} e^{-c\|x\|^2} dx \\ &= \int G_t * |f|(x) e^{-c\|x\|^2} dx \quad (\text{where } * \text{ is convolution}) \\ &= \left(\frac{\pi}{c}\right)^{k/2} \int |f(x)| G_t * G_{1/c}(x) dx \\ &= \left(\frac{\pi}{c}\right)^{k/2} \int |f(x)| G_{t+1/c}(x) dx < \infty. \end{aligned}$$

So T_V^t takes $\mathcal{M}$ to $\mathcal{M}$. Finally:

$$\begin{aligned} T_V^s T_V^t f(x) &= E_x \left\{ \exp \left[- \int_0^s V(\xi_r) dr \right] T_V^t f(\xi_s) \right\} \\ &= E_x \left\{ \exp \left[- \int_0^s V(\xi_r) dr \right] E_{\xi_s} \left\{ \exp \left[- \int_0^t V(\xi_u) du \right] f(\xi_t) \right\} \right\} \\ &= E_x \left\{ \exp \left[- \int_0^s V(\xi_r) dr \right] \exp \left[- \int_0^t V(\xi_{u+s}) ds \right] f(\xi_{t+s}) \right\}, \end{aligned}$$

by the strong Markov property. This can be rewritten as $T_V^{s+t} f(x)$.

Next, we examine the effect of varying V .

THEOREM 3.2. $V \geq W$ a.e. $\Rightarrow T_V^t f \leq T_W^t f$ for all a.e. nonnegative f in $\mathcal{M}$. Furthermore, if $V_n \uparrow V$ a.e., then $T_{V_n}^t f \downarrow T_V^t f$ for such f .

Proof. The only difficulty is to see that if $V = W$ a.e., then $T_V^t f = T_W^t f$. This is true because, for any set S of measure 0 in $\mathcal{X}$, we have

$$E_x \left\{ \int_0^t 1_S(\xi_s) ds \right\} = \int_0^t \int_S G_x(y-x) dy ds = 0,$$

so that V is unaffected by a change on a set of measure 0.

Finally, we consider the action of T_V^t on the various $\mathcal{L}_p$ spaces over $\mathcal{X}$ (taken with Lebesgue measure, normalized via the inner product in $\mathcal{X}$), and also on the space $\mathcal{B}$ of bounded Borel functions on $\mathcal{X}$. By $\|\cdot\|_p$ we will mean the norms in $\mathcal{L}_p = \mathcal{L}_p(\mathcal{X})$, $1 \leq p \leq \infty$, and by just plain $\|\cdot\|$ the norm in $\mathcal{B}$.

THEOREM 3.3. T_V^t is a contraction on $\mathcal{L}_p$ ($1 \leq p \leq \infty$) and on $\mathcal{B}$. Furthermore, if $f \in \mathcal{L}_p$ and $g \in \mathcal{L}_q$, with $1/p + 1/q = 1$, then $\int T_V^t f(x) g(x) dx = \int f(x) T_V^t g(x) dx$.

Proof. Since $T_V^t f(x) \leq T_0^t f(x)$ for each nonnegative f in $\mathcal{M}$, the first sentence will follow once we have it for the case $V = 0$. But this case is well known for $p = 1$ and $p = \infty$, while for other p it follows from the Riesz convexity theorem.

As for the self-adjointness: Consider $\Omega \times \Omega$. If we denote by $\tilde{\Omega}$ the subset of pairs (ω, ω') such that $\xi_0(\omega) = \xi_0(\omega')$, then $\tilde{\Omega}$ can be identified with the space of all continuous functions $\tilde{\omega}$ from the real line to $\mathcal{X}$, by letting

$$\tilde{\omega}(t) = \begin{cases} \omega(-t) & \text{if } t \leq 0 \\ \omega'(t) & \text{if } t \geq 0. \end{cases}$$

The measure $\text{Pr}_x \times \text{Pr}_x$ has its support on $\tilde{\Omega}$. We set $\tilde{\text{Pr}}\{\tilde{\Lambda}\} = \int \text{Pr}_x \times \text{Pr}_x\{\tilde{\Lambda}\} dx$, for $\tilde{\Lambda}$ a measurable set in $\tilde{\Omega}$. $\tilde{\text{Pr}}$ is, of course, an infinite measure. Define $\tilde{\xi}_t(\tilde{\omega}) = \tilde{\omega}(t)$. Then it is easy to see that the joint distributions of $\tilde{\xi}_{t_1}, \dots, \tilde{\xi}_{t_n}$ are the same as those of $\tilde{\xi}_{t_1+h}, \dots, \tilde{\xi}_{t_n+h}$, and of $\tilde{\xi}_{-t_1}, \dots, \tilde{\xi}_{-t_n}$. Thus,

$$\begin{aligned} (T_V^t f, g) &= \int E_x \left\{ \exp \left[- \int_0^t V(\xi_s) ds \right] f(\xi_t) \right\} g(x) dx \\ &= \tilde{E} \left\{ \exp \left[- \int_0^t V(\tilde{\xi}_s) ds \right] f(\tilde{\xi}_t) g(\tilde{\xi}_0) \right\} = \tilde{E} \left\{ \exp \left[- \int_0^t V(\tilde{\xi}_{t-s}) ds \right] f(\tilde{\xi}_0) g(\tilde{\xi}_t) \right\} \\ &= E \left\{ \exp \left[- \int_0^t V(\tilde{\xi}_u) du \right] f(\tilde{\xi}_0) g(\tilde{\xi}_t) \right\} = \int E_x \left\{ \exp \left[- \int_0^t V(\xi_u) du \right] g(\xi_t) \right\} f(x) dx \\ &= (f, T_V^t g). \end{aligned}$$

REMARK. The self-adjointness has been proved by Gettoor for $\mathcal{L}_2$, in [8], and could be shown generally by approximation.

THEOREM 3.4. If f is in $\mathcal{L}_p$ and $1/q + 1/p = 1$, then $\|T_V^t f\| < C(q)t^{l(q)}\|f\|_p$, where $C(q) = \pi^{l(q)} q^{-k/2q}$ and $l(q) = (1/q - 1)k/2$. Furthermore, if S_N is the

N -sphere, then $\|I_{S_N^+} T_V^t f\| \rightarrow 0$ as $N \rightarrow \infty$, uniformly in V and in t restricted to an interval $0 < t_0 < t < t_1 < \infty$.

NOTATION. $S^\perp$ is the complement of the set S .

Proof. $|T_V^t f(x)| \leq |G_t * f(x)| \leq \|G_t\|_q \|f\|_p$. Evaluating $\|G_t\|_q$ gives the first part.

Choose $\varepsilon > 0$. Choose M so big that $\|I_{S_M^+} f\|_p < \varepsilon(2\|G_{t_0}\|_q)^{-1}$. Then $\|T_V^t I_{S_M^+} f\| < \varepsilon/2$. Now, if $\|y\| \leq M$ and $\|x\| \geq N$, then $\|y - x\| \geq N - M$, and $|G_t(y - x)| \leq (\pi t_0)^{-k/2} e^{-(N-M)^2/t_1}$ provided $t_0 \leq t \leq t_1$. Now, $I_{S_M^+} f$ is an $\mathcal{L}_p$ function with support of finite measure, hence an $\mathcal{L}_1$ function. Choose N so large that $|G_t(y - x)| < \varepsilon(2\|I_{S_M^+} f\|_1)^{-1}$ if $\|y\| \leq M$, $\|x\| \geq N$, and $t_0 \leq t \leq t_1$. Then $|T_V^t f(x)| < \varepsilon$ if $\|x\| \geq N$ and $t_0 \leq t \leq t_1$.

LEMMA. If V_0 is in $\mathcal{L}_p$ for some $p > k/2$, then

$$E_x \left\{ \int_0^r V_0(\xi_u) du \right\} \leq c(p) \|V_0\|_p \cdot r^{k/2+1}.$$

Proof.

$$\begin{aligned} \left| E_x \left\{ \int_0^r V_0(\xi_u) du \right\} \right| &= \left| \int_0^r V_0(x+y) G_u(y) dy du \right| \\ &\leq \int_0^r \|V_0\|_p \|G_u\|_q du = \|V_0\|_p \int_0^r C(q) u^{l(q)} du \end{aligned}$$

(using the notation in the proof of the previous theorem)

$$= c(p) \|V_0\|_p r^{k/2+1},$$

where $c(p) = C((1 - 1/p)^{-1})(1 - k/2p)^{-1}$.

THEOREM 3.5. If, for some $\bar{p} > k/2$, V is in $\mathcal{L}_{\bar{p}}$ on an open set $\mathcal{O}$, and f is in any $\mathcal{L}_p$ class, then $T_V^t f$ is continuous in $\mathcal{O}$. More precisely: for any compact subset C of $\mathcal{O}$, $\varepsilon > 0$, and $t_1 > 0$, we can choose r_0 such that $T_V^t f(x)$ differs by less than ε from the continuous function $T_0^r T_V^{t-r} f(x) = G_r * T_V^{t-r} f(x)$, for all x in C , $0 < r \leq r_0$, and $t \geq t_1$.

Proof. Let $\Lambda_s = \{\omega \mid \xi_r(\omega) \in \mathcal{O} \text{ for } 0 \leq r \leq s\}$. This is in $\mathcal{F}$, as is also the set $\Gamma_{s,x} = \{\omega \mid \int_0^s V(\xi_r) dr < \alpha\}$. Write Φ_r^s for the function $\exp[-\int_r^s V(\xi_u) du]$ on Ω . Then

$$\begin{aligned} T_V^t f(x) &= E_x \{\Phi_0^t f(\xi_t)\} = E_x \{\Lambda_s^\perp \cup \Gamma_{r,\omega}^\perp (\Phi_0^r - 1) \Phi_r^t f(\xi_t)\} \\ &\quad + E_x \{\Lambda_s \cap \Gamma_{r,\omega} (\Phi_0^r - 1) \Phi_r^t f(\xi_t)\} + T_0^r T_V^{t-r} f(x). \end{aligned}$$

We estimate the first two terms.

Let C be a compact subset of $\mathcal{O}$, d its distance from $\mathcal{O}^\perp$, and x any point in C . Then $\Pr_x\{\Lambda_s^\perp\} \leq \Pr_0\{\|\xi_u\| \geq d \text{ for some } u \text{ in } [0, s]\}$. This is known to go to 0 as $s \downarrow 0$. Thus, we can choose s so small that $\Pr_x\{\Lambda_s^\perp\}C(q)(t_1/2)^{l(q)} < \varepsilon/3$ ($l(q)$ defined as in Theorem 3.4). We can also require that $s < t_1/2$.

Next, denote by V_0 the function $I_\sigma V$. Then if $r \leq s$,

$$\Pr_x\{\Lambda_s \cap \Gamma_{\alpha, r}^\perp\} = \Pr_x\left\{\Lambda_s, \int_0^r V_0(\xi_u) du \geq \alpha\right\} \leq \Pr_x\left\{\int_0^r V_0(\xi_u) du \geq \alpha\right\}.$$

From the previous lemma, this is dominated by $c(\bar{p})/\alpha \|V_0\|_{p^r}^{-k/2+1}$. If α is pre-assigned, then by choosing r sufficiently small (and in particular, smaller than s and $t_1/2$), we can thus guarantee that

$$\Pr_x\{\Lambda_s \cap \Gamma_{\sigma, r}^\perp\}C(q)(t_1/2)^{l(q)} < \varepsilon/3.$$

α was at our disposal. We choose it so small that $(1 - e^{-\alpha})C(q)(t_1/2)^{l(q)} < \varepsilon/3$. Then

$$\begin{aligned} |E_x\{\Lambda_s \cap \Gamma_{r, \alpha}(\Phi_0^r - 1)\Phi_r^t f(\xi_t)\}| &\leq (1 - e^{-\alpha})E_x\{\Phi_r^t |f|(\xi_t)\} \leq (1 - e^{-\alpha})E_x\{T_V^{t-r} |f|(\xi_r)\} \\ &\leq (1 - e^{-\alpha})C(q)(t - r/2)^{l(q)} \|f\|_p \leq \varepsilon/3 \|f\|_p. \end{aligned}$$

Thus, for all $t \geq t_1$ and x in C , $\|T_V^t f - T_0^r T_V^{t-r} f\|$ is dominated by $\varepsilon \|f\|_p$, the choice of r depending on ε , t_1 and $\|V_0\|_{\bar{p}}$.

Summarizing some of this:

COROLLARY 3.1. T_V^t takes each $\mathcal{L}_p$ class into $\mathcal{L}_p \cap \mathcal{B}_0$ (where $\mathcal{B}_0$ is $\{f \text{ in } \mathcal{B} | f(x) \rightarrow 0 \text{ as } \|x\| \rightarrow \infty\}$). Furthermore, $T_V^t f$ is continuous on the open set $\{x | V \text{ is in } \mathcal{L}_p \text{ in some neighborhood of } x, \text{ for some } p > k/2\}$.

REMARK 3.1. Choosing V to be $+\infty$ on the complement of some set is one way of relativizing the process to that set (see also the method used by Gettoor in [9], where everything gets cut down to an open set G).

4. The infinitesimal generator. In [8] Gettoor incorrectly said that T_V^t is continuous at 0 as a semigroup on $\mathcal{L}_2$ (and hence, has a densely defined infinitesimal generator). This statement was, however, corrected in [9], and even in [8] he mentioned a necessary and sufficient condition on V that T_V^t be continuous in this sense. Also, in [9], a rather stringent sufficient condition is given. The condition in [8] is just that $\lim_{t \rightarrow 0} \exp[-\int_0^t V(\xi_s) ds] = 1$ $\Pr_x$ -a.e., for almost every x in $\mathcal{X}$. We shall not assume this, but rather investigate for arbitrary V the subspace on which T_V^t is continuous; or, equivalently, the closure of the domain of the infinitesimal generator of T_V^t .

Consider, for fixed x , the condition that $\lim_{t \rightarrow 0} \exp[-\int_0^t V(\xi_s) ds] = 0$ $\Pr_x^\sigma$ -a.e. This condition is actually independent of σ . One way of seeing this is the following. Recall the map j_σ from Ω to Ω sending ω to the function whose value at

t is $\omega(t\sigma)$. Then j_σ is a 1-1 measure-preserving transformation from $(\Omega, \mathcal{F}, \text{Pr})$ to $(\Omega, \mathcal{F}, \text{Pr}^\sigma)$, and $\int_0^t V(\xi_s + x)ds = \sigma \int_0^t V(\xi_s \cdot j_\sigma + x)ds$.

Let $\Omega_x = \{\lim_{t \rightarrow 0} \exp[-\int_0^t V(\xi_s + x)ds] = 1\}$. This is exactly

$$\left\{ \lim_{t \rightarrow 0} \int_0^t V(\xi_s + x)ds = 0 \right\},$$

and also $\{\exists t > 0 \text{ such that } \int_0^t V(\xi_s + x)ds < \infty\}$. The set is in $\mathcal{F}_t$ for each $t > 0$. Thus, by the zero-one law, it differs from a set in $\mathcal{F}_0$ by a set which has Pr^σ -measure 0 for each σ . Then for given x , Ω_x either has Pr^σ -measure 0 for all σ or has Pr^σ -measure 1 for all σ .

V will be called *controllable* at x if $\text{Pr}\{\Omega_x\} = 1$. So if V is *not* controllable at x , then for each moderate f we have $T_V^t f(x) = 0$ for all $t > 0$.

DEFINITION. Let C_V be the set of points where V is controllable, and let I_V be the operation of multiplication by the function which is 1 on C_V and 0 elsewhere. Also, let $\mathcal{L}_p^V$ be $\mathcal{L}_p$ with respect to Lebesgue measure cut down to C_V , with corresponding norms $\|\cdot\|_p^V$.

THEOREM 4.1. (a) For each moderate f we have

$$I_V T_V^t I_V f = T_V^t f \text{ a.e.},$$

and

$$\lim_{t \downarrow 0} T_V^t f = I_V f \text{ a.e.}$$

(b) Furthermore, T_V^t is strongly continuous on each $\mathcal{L}_p^V$, $1 \leq p < \infty$, and is weak $*$ continuous on $\mathcal{L}_\infty^V$. Let A_V be the infinitesimal generator (no distinction is necessary for different p , since there is agreement on functions in several different $\mathcal{L}_p^V$).

(c) On $\mathcal{L}_2^V$, A_V is a nonnegative self-adjoint operator, and $e^{-tA_V} = T_V^t$.

Proof. If x is not in C_V , then $E\{\exp[-\int_0^t V(\xi_s + x)ds]f(\xi_t + x)\} = 0$ for each $t > 0$. So $I_V T_V^t f = \text{a.e. } T_V^t f$ for moderate f . If f is actually in $\mathcal{L}_2$, then self-adjointness of T_V^t tells us that $I_V T_V^t I_V f = \text{a.e. } T_V^t f$. For general moderate f , the last equality still holds, by the Lebesgue convergence theorem. The fact that $\lim_{t \downarrow 0} T_V^t f(x) = f(x)$ for a.e. x in C_V can be shown as follows: first one proves it for continuous f , by applying the Lebesgue convergence theorem; then for arbitrary moderate f by Theorem 3.1.

As for (b): since $T_V^t f$ converges a.e. to $I_V f$ as $t \downarrow 0$, and since $\|T_V^t f\|_p^V \leq \|T_V^t f\|_p = \|T_V^t I_V f\|_p \leq \|I_V f\|_p = \|f\|_p^V$, we have that $T_V^t f \rightarrow f$ in $\mathcal{L}_p^V$, in the weak topology, or, in the case $p = \infty$ in the weak $*$ topology. Since each T_V^t is a contraction, we get T_V^t weakly continuous at all $t \geq 0$ if $1 \leq p < \infty$, or weak $*$ continuous if $p = \infty$. If $1 \leq p < \infty$, then $\mathcal{L}_p^V$ is separable, so weak continuity implies strong measurability by [10], Theorem 3.55, and therefore T_V^t is strongly continuous for $t \geq 0$ by [10, Theorem 10.5.5].

(c) Finally, that A_V on $\mathcal{L}_2^V$ is a nonnegative self-adjoint operator can be seen as follows. T_V^t has some representation of the form e^{-tB} for a self-adjoint operator B (easily seen to be positive), by [13, XI, 2]. Now, the domain of A_V is the range of $\int_0^\infty T_V^t e^{-\lambda t} dt = 1/(B + \lambda)$, and $(A_V + \lambda)(1/(B + \lambda))f = f$. Thus, A_V is precisely B .

Next, two theorems which cast a little light on the question "what is C_V for given V ?"

THEOREM 4.2. C_V is a.e. contained in the set where V is finite.

Proof. Let $R = \{x \mid V(x) = \infty\}$. Then, for a.e. x in R , the set R has density 1 at x . Selecting such an x :

$$E_x \left\{ \frac{1}{t} \int_0^t 1_R(\xi_s) ds \right\} = \frac{1}{t} \int_0^t E_x \{ 1_R(\xi_s) \} ds = \frac{1}{t} \int_0^t G_s * 1_R(x) ds.$$

Now, $G_s * 1_R(x) \rightarrow 1$ as $s \rightarrow 0$, since G_s is an approximate identity and R has density 1 at x . Thus

$$\frac{1}{t} \int_0^t G_s * 1_R(x) ds \rightarrow 1.$$

As a consequence, $\Pr_x \{ \int_0^t V(\xi_s) ds = \infty \} = 1$ for each $t > 0$, and x is in R .

THEOREM 4.3. If $p > k/2$ and $V \in \mathcal{L}_p(\mathcal{O})$, $\mathcal{O}$, open $\subset K$, then $C_V \supset \mathcal{O}$ a.e.

Proof. It is no loss of generality to assume V vanishes outside $\mathcal{O}$, since each path starting at an x in $\mathcal{O}$ stays there for a while. But then the lemma after Theorem 3.4 tells us that $E_x \{ \int_0^t V(\xi_s) ds \} < \infty$, so that $\int_0^t V(\xi_s) ds < \infty$ $\Pr_x$ -a.e., and therefore x is in C_V if it is in $\mathcal{O}$.

REMARK 4.1. Operators on $\mathcal{L}_p^V$ are in an obvious 1-1 correspondence with operators B on $\mathcal{L}_p$ such that $I_V B I_V = B$. Thus, we will occasionally treat A_V as an operator on $\mathcal{L}_p$, without further comment.

REMARK 4.2. If $V = 0$, then A_V is just the negative of the usual Laplacian (on $\mathcal{L}_2$). More generally, it is shown in [8] that if C_V is almost all of $\mathcal{X}$, and M_V is the operation of multiplication by V on $\mathcal{H}$, then

$$A_V \supset (-\Delta + M_V) \big|_{\mathcal{D}_\Delta \cap \mathcal{D}_{M_V}} \quad (\text{where } \mathcal{D}_T \text{ is the domain of } T).$$

For example, if V is bounded, then A_V is just $-\Delta + M_V$. However, it would be of interest to have the answer to the following question, for instance. Suppose the Laplacian of f exists locally, in some sense, but the function g thereby obtained is no longer in $\mathcal{L}_2$. Suppose, however, that $-g + Vf$ is in $\mathcal{L}_2$. Is it then the case that f is in $\mathcal{D}_{A_V}$ and $A_V f = -g + Vf$? (The considerations of [9, §4], do not apply, unfortunately, because Δ is not a "local operator" in the sense used there.)

REMARK 4.3. Here is a phenomenon which was surprising at least to me. Recall that

$$T_0^t = e^{-t\Delta}f = G_t * f$$

is actually infinitely differentiable for all moderate f . Recall also that for general V , and f in $\mathcal{L}_p$, T_V^t is continuous where V is in some $\mathcal{L}_{\bar{p}}$ class (Corollary 3.1). One might therefore expect that if, say, V were bounded then $T_V^t f$ would be infinitely differentiable. However, this is far from true!

EXAMPLE. Let V be a nonnegative bounded measurable function, and let f be in $\mathcal{D}_{A_V}$. Thus $T_V^t f$ is again in $\mathcal{D}_{A_V}$, and $A_V T_V^t f = T_V^t A_V f$. But $\mathcal{D}_{A_V} = \mathcal{D}$ and $A_V = -\Delta + M_V$, so $-\Delta T_V^t f + V T_V^t f = A_V T_V^t f = T_V^t A_V f$. Now, $T_V^t f$ is continuous, as is also $T_V^t A_V f$, by Corollary 3.1. If $T_V^t f$ had two continuous derivatives, then a representative for $T_V^t f$ could be chosen which was continuous. But $V T_V^t f$ can be made as irregular as one likes, for example by choosing V to be unequal to any continuous function on any open set. So $T_V^t f$ cannot always be twice continuously differentiable, even if V is bounded and f itself is a C_∞ function with compact support. (However, regularity assumptions on V would presumably result in regularity for $T_V^t f$.)

5. Complexification of the semigroup, and the limiting Feynman integral. Let Λ be the set of complex numbers with positive real part, $\bar{\Lambda}$ its closure. Let A be a fixed nonnegative self-adjoint operator on the Hilbert space $\mathcal{H}$. For any ζ in $\bar{\Lambda}$, the functional calculus defines a bounded operator $e^{-\zeta A}$. This operator is unitary if ζ is imaginary, nonnegative and self-adjoint if ζ is non-negative, and has norm ≤ 1 for ζ in $\bar{\Lambda}$. The map $\zeta \rightarrow e^{-\zeta A}$ is continuous in the strong operator topology for ζ in $\bar{\Lambda}$, and satisfies $e^{-\zeta A} e^{-\zeta' A} = e^{-(\zeta+\zeta')A}$. For ζ in Λ , it is continuous in the uniform operator topology, and even holomorphic. These facts are all, at worst, straightforward applications of the functional calculus.

EXAMPLE. We can extend T_V^t , as an operator on $\mathcal{L}_2^V$, to T_V^ζ for ζ in $\bar{\Lambda}$, by setting $A = A_V$.

We quote, for later use, a fact about convergence of analytic functions.

FACT 5.1 (VITALI). Let $F_1, F_2, \dots$ be a sequence of analytic functions on Λ , with values in a Banach space. Suppose the F_n are uniformly bounded in norm on each compact subset of Λ . Suppose also that they converge in norm at all points of $(0, \infty)$. Then they converge in norm on Λ , uniformly on compact subsets, to an analytic function F_∞ on Λ .

Proof. [10, p. 104, Theorem 3.14.1].

For the purposes of our first theorem, we will want, for each $t > 0$, $\Pr_x\{V(\xi_s) \text{ Riemann-integrable on } (0, t)\} = 1$ for a.e. x . This amounts to $\Pr_x\{V(\xi_s) \text{ bounded and a.e. continuous on } (0, t)\} = 1$ for a.e. x . Call such a V Riemann-approximable. Observe that if V is Riemann-approximable then C_V is almost all of E , so that T_V^t is strongly continuous at 0 in $\mathcal{L}_p$.

THEOREM 5.1. Suppose V is Riemann-approximable. Let τ be a finite sequence of positive numbers: $\tau = (\tau_1, \dots, \tau_n(\tau))$, with $\sum \tau_j = 1$. Let $|\tau| = \max_j \tau_j$. For ζ in Λ , let $T_{V, \tau}^\zeta = \prod_j e^{-\zeta \tau_j V} e^{\zeta \tau_j \Delta}$, everything operating on $\mathcal{L}_2$. This is clearly holomorphic on Λ , strongly continuous on $\bar{\Lambda}$. Then $\lim_{|\tau| \rightarrow 0} T_{V, \tau}^\zeta$ exists in the strong operator topology, and uniformly for ζ in any compact subset of Λ , and equals T_V^ζ . Finally: if ϕ is any integrable function on the real line, then

$$\lim_{|\tau| \rightarrow 0} \int_{-\infty}^{+\infty} (T_{V, \tau}^{is} f, g) \phi(s) ds = \int_{-\infty}^{+\infty} (T_V^{is} f, g) \phi(s) ds,$$

for all f, g in $\mathcal{L}_2$ (where T_V^{is} is the strongly continuous extension of T_V^ζ to the imaginary axis).

REMARK 5.1. The fact of convergence was proved by D. Babbitt [1], under the added assumption that V satisfied a local Lipschitz condition. The proof of the present generalization is just a simplification of Babbitt's proof.

Proof of theorem. Consider the sum $\sum_j V(\xi_{(\tau_1 + \dots + \tau_j)t}(\omega)) \tau_j t$. This is a Riemann sum for the integral $\int_0^t V(\xi_s(\omega)) ds$, using the partition $(\tau_1 t, \dots, \tau_n(\tau) t)$. Thus, $\sum_j V(\xi_{(\tau_1 + \dots + \tau_j)t}(\omega)) \tau_j t$ converges to $\int_0^t V(\xi_s(\omega)) ds$ as $|\tau| \rightarrow 0$, for Pr_x -almost every ω . Now let f be in $\mathcal{M}$. Then, since the functions within $E_x\{\dots\}$ are all bounded in norm by $|f(\xi_t)|$, and converge Pr_x -a.e., we have

$$\lim_{|\tau| \rightarrow 0} E_x \left\{ \exp \left[- \sum_j V(\xi_{(\tau_1 + \dots + \tau_j)t}) \tau_j t \right] f(\xi_t) \right\} = T_V^t f(x)$$

for each x . Also

$$\left| E_x \left\{ \exp \left[- \sum_j V(\xi_{(\tau_1 + \dots + \tau_j)t}) \tau_j t \right] f(\xi_t) \right\} - T_V^t f(x) \right|^2 \leq 2 E_x \{ |f(\xi_t)|^2 \}.$$

Thus, if f is in $\mathcal{L}_2$, then $\lim_{|\tau| \rightarrow 0} \|T_{V, \tau}^t f - T_V^t f\|_2 = 0$, i.e. $T_{V, \tau}^t$ converges strongly to T_V^t . Now we can apply Fact 4.1 to get the existence of a holomorphic limit T_V^ζ , ζ in Λ , which must agree with $e^{-\zeta \Delta V}$ on $\bar{\Lambda}$ since it agrees for $\zeta > 0$.

The fact that $\lim_{|\tau| \rightarrow 0} \int_{-\infty}^{\infty} (T_{V, \tau}^{is} f, g) \phi(s) ds = \int_{-\infty}^{\infty} (T_V^{is} f, g) \phi(s) ds$ for all ϕ in $\mathcal{L}_1$ is a consequence of the fact that $(T_{V, \tau}^\zeta f, g)$ and $(T_V^\zeta f, g)$ are bounded holomorphic functions and $(T_{V, \tau}^\zeta f, g) \rightarrow (T_V^\zeta f, g)$ on Λ . This can be seen as follows. Let $P_t(s) = (1/\pi)(t/(t^2 + s^2))$. P_t is an approximate identity, so that $P_t * \phi \rightarrow \phi$ in $\mathcal{L}_1(-\infty, \infty)$. If Ψ is any bounded analytic function in the right half plane, then $\Psi(t + is) = \int P_t(s - s') \Psi(is') ds'$. Now let $\Psi_\tau(\zeta) = (T_{V, \tau}^\zeta f, g)$, and $\Psi(\zeta) = (T_V^\zeta f, g)$. Notice that $\int P_t * \psi(s) \phi(s) ds = \int \psi(s) P_t * \phi(s) ds$. Thus: $\int (\Psi_\tau(is) - \Psi(is)) \phi(s) ds = \int (\Psi_\tau(t + is) - \Psi(t + is)) \phi(s) ds + \int (\Psi_\tau(is) - \Psi(is)) (\phi(s) - P_t * \phi(s)) ds$. The second term has absolute value $\leq \text{const.} \int |\phi(s) - P_t * \phi(s)| ds$. By choosing t small, this can be made arbitrarily small (for fixed ϕ). The first term can then be made small by choosing $|\tau|$ small, since $|\Psi_\tau(t + is) - \Psi_\tau(t + is)|$ stays bounded by $2 \|f\| \|g\|$, and converges to 0 for each s .

REMARK 5.2. Observe that one point which came out in the proof was that for each x at which V is Riemann-approximable,

$$E_x\{\exp[-\sum_j V(\xi_{(\tau_1+\dots+\tau_j)t})\tau_j t]f(\xi_t)\}$$

converges to $T_V^t f(x)$, for each f in $\mathcal{M}$. (Note: Riemann-approximability at x has not been defined, but it should be obvious what is meant.)

What sort of V are Riemann-approximable? A large class is the following. It permits arbitrarily bad infinities on a set of capacity zero.

THEOREM 5.2. *Let D be the closed set of x for which V is essentially unbounded in every neighborhood of x . Suppose D forms a set of capacity 0. Suppose also the points of discontinuity of V form a set of measure 0. Then V is Riemann-approximable.*

Proof. By changing V on a set of measure 0 in $D^\perp$, we can assume that V is actually locally bounded in $D^\perp$. Namely, let $C_n \uparrow D$, C_n compact, and replace V on $C_n - C_{n-1}$ by $V \wedge \|1_{C_n} V\|$. This will not introduce any new discontinuities. Then $\Pr_x\{\xi_s \text{ lies in } D \text{ for some } s\} = 0$, for a set D of capacity 0. See, for example, [4]. Thus, for $\Pr_x$ -a.e. ω , the range of $\xi_s(\omega)$, $0 \leq s \leq t$, is a compact subset of $D^\perp$, and so $s \rightarrow V(\xi_s(\omega))$ is bounded on $[0, t]$. Also, for $\Pr_x$ -a.e. ω , the set of s for which $\xi_s(\omega)$ lies in the set of discontinuities of V has Lebesgue measure 0. Thus, for $\Pr_x$ -a.e. ω , $\xi_s(\omega)$ is Riemann-integrable for $0 < s \leq t$.

6. The Green's function. Recall that if $f \in \mathcal{L}_p$ then $\|T_V^t f\|_\infty \leq C(q)t^{(1/q-1)^{k/2}}$. So, for $1 \leq p < \infty$, $T_V^t f(x) = \int k_x^t(y)f(y)dy$, where k_x is an equivalence class of Lebesgue measurable functions, and $\|k_x^t\|_q \leq C(q)t^{(1/q-1)^{k/2}}$. For $f \geq 0$ in $\mathcal{L}_\infty$ choose $f_n \in \mathcal{L}_2$, $f_n \uparrow f$. Then $T_V^t f_n(x) \uparrow T_V^t f(x)$, so that

$$\int k_x^t(y)f(y)dy = \lim_{n \rightarrow \infty} \int k_x^t f_n(y)dy = \lim_{n \rightarrow \infty} T_V^t f_n(x) = T_V^t f(x),$$

so again we have $\int k_x^t(y)f(y)dy = T_V^t f(x)$, $\|k_x^t\|_1 \leq C(1)$, independent of t .

We introduce a canonical version of k_x^t .

LEMMA 6.1. $\int k_x^r(z)k_y^s(z)dz$ is, for each x , equal to $k_x^t(y)$ for almost every y , provided $r + s = t$. Further, it is independent on the choice of r and s .

Proof. $k_x^s(y)$ can be chosen a jointly Borel measurable function of x and y , since the map $x \rightarrow k_x^s$ is a measurable map from $\mathcal{X}$ to, for example, $\mathcal{L}_2$. Furthermore,

$$\int g(x)k_x^s(y)f(y)dy = \int g(x)T_V^s f(x)dx = \int T_V^s g(y)f(y)dy,$$

since T_V^s is self-adjoint. So

$$\begin{aligned} \int \left(\int k_x^r(z)k_z^s(y)dz \right) f(y)dy &= \int \int k_x^r(z)k_z^s(y)f(y)dydz = \int k_x^r(z)T_V^s f(z)dz = T_V^r T_V^s f(x) \\ &= \int k_x^t(y)f(y)dy. \end{aligned}$$

To show independence of r and s , we choose r, s, r', s' , with $r + s = r' + s'$. Assume $r < r'$. Then, from what has been shown,

$$\begin{aligned} \int k_x^{r'}(z)k_y^{s'}(z)dz &= \int k_x^{r'+(r'-r)}(z)k_y^{s'}(z)dz \\ &= \int \int k_x^r(q)k_w^{r'-r}(z)k_z^{s'}(y)dw dz \\ &= \int \int k_x^r(w)k_z^{s-s'}(w)k_z^{s'}(y)dw dz \\ &= \int k_x^r(w)k_w^s(y)dw, \end{aligned}$$

which completes the proof.

Now it makes sense to define $K_V^t(x, y) = \int k_x^r(z)k_y^s(z)dz$, since it is independent of r and s , provided $r + s = t$.

Thus we have

REMARK 6.1. (a) There is a function $K_V^t(x, y)$ such that $T_V^t f(x) = \int K_V^t(x, y)f(y)dy$ for f in any $\mathcal{L}_p$ -class. K_V^t is symmetric. Further, $K_V^{s+t}(x, z) = \int K_V^s(x, y)K_V^t(y, z)dy$.

(b) The last property, together with the fact that $T_V^t f(x) = \int K_V^t(x, y)f(y)dy$ for enough f , uniquely determine K_V^t . (I use the label "remark" rather than "theorem" in order to avoid being precise about the word "enough".)

Properties of T_V^t easily translate into properties of K_V^t . For example:

THEOREM 6.1. $x \rightarrow K_V^t(x, \cdot)$ is continuous into all $\mathcal{L}_p$, and K_V^t is jointly continuous in x and y , on the open set where V is locally integrable.

Proof. The first statement is an immediate consequence of the definition and of Theorem 3.5. As for the second part: if $x_n \rightarrow x$ and $y_n \rightarrow y$, and V is locally integrable at x and y , then

$$\begin{aligned} &\left| \int K_V^t(x_n, z)K_V^s(y_n, z)dz - \int K_V^t(x, z)K_V^s(y, z)dz \right| \\ &\leq \|K_V^t(x_n, \cdot) - K_V^t(x, \cdot)\|_2 \|K_V^s(y_n, \cdot)\|_2 + \|K_V^t(x, \cdot)\|_2 \|K_V^s(y_n, \cdot) - K_V^s(y, \cdot)\|_2. \end{aligned}$$

But $\|K_V^s(y_n, \cdot)\|_2$ and $\|K_V^t(x_n, \cdot)\|_2$ stay bounded, while the other factors go to zero.

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